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An $\mathcal{N} = 1$ Triality by Spectrum Matching

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Abstract

On promoting the type IIA side of the $\mathcal{N} = 1$ Heterotic/type IIA dual pairs of [1] to M-theory on a 'barely G_2 Manifold' of [2], by spectrum-matching we show a possible triality between Heterotic on a self-mirror Calabi-Yau, M-theory on the above 'barely G_2-Manifold' constructed from the Calabi-Yau on the type IIA side and F-theory on an elliptically fibered Calabi-Yau 4-fold fibered over a trivially rationally ruled $\mathbf{CP}^1 \times \mathcal{E}$ base, $\mathcal{E}$ being the Enriques surface. There are some interesting properties of the antiholomorphic involution used in [1] for constructing the type IIA orientifold and by us in constructing the 'barely G_2 manifold', that we also study.

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1 Introduction

As $\mathcal{N}=1$ supersymmetry in four dimensions is of phenomenological interest, it is important to understand possible dualities between different ways of arriving at the same amount of supersymmetry via suitable compactifications. In this regard, the results of [1, 3, 4, 5] are of particular interest. While [1] construct such string dual pairs, [3, 4, 5] also give $\mathcal{N}=1$ Heterotic/M-theory dual pairs. As M-theory on G_2-holonomy manifolds gives $\mathcal{N}=1$ supersymmetry, especially after explicit examples of the same (and Spin(7)) in [6, 7], exceptional holonomy compactifications of M-theory becomes quite relevant for the above purpose. In the literature, so far, the $\mathcal{N}=1, D=4$ Heterotic/M-theory dual pair constructions, stem one way or the other from the Heterotic on T^3 and M-theory on $K3$ $D=7$ duality [8, 9]. Lets elaborate this point a little. Assuming the 'adiabatic argument' of [1], the duality between heterotic on a CY that is a T^3-fibration over a 3-fold and M-theory on a G_2-holonomy manifold that is a $K3$-fibration over the same 3-fold of [4, 5] was obtained by fiberwise application of duality with a twist (to get $\mathcal{N}=1$ supersymmetry) to the abovementioned $D=7$ Heterotic/M-theory duality. Similarly, by application of fiberwise duality, accompanied by a twist, with the 3-fold base above being a T^3, the heterotic/M-theory dual pair of [3] was obtained. The 7-manifold on the M-theory side in [3] will be the motivation for the choice of the 7-manifold relevant to us. Now, lets come to string-string duality in $D=6$: Heterotic on T^4 is dual to type IIA on $K3$. By fiberwise application of duality using a common $\mathbf{CP}^1$ base, and taking $K3$ to be elliptically fibered (T^2-fibration over $\mathbf{CP}^1$) and the CY to be a $K3$-fibration over $\mathbf{CP}^1$, $\mathcal{N}=2$ dual pairs of heterotic on $K3\times T^2$ with vector bundles on $K3$ embedded in $E_8\times E_8$, and type IIA on the $K3$-fibered Calabi-Yau were constructed in [10]. From these dual pairs, $\mathcal{N}=1$ dual pairs were obtained in [1] by orientifolds of the CY on the type IIA side whose image on the heterotic side corresponded to the self-mirror CY of [11]. The question is what the $\mathcal{N}=1$ Heterotic/M-theory analog of the Heteoric/type IIA $\mathcal{N}=1$ dual pair of [1] is. As the $D=7$ Heterotic/M-Theory duality is related to the $D=6$ Heterotic/String duality (as the decompactification limit - see [9]), it is reasonable to think that there has to be such an $\mathcal{N}=1$ Heterotic/M-theory dual pair. Additionally, it will be interesting to work out an example that is able to explicitly relate an $\mathcal{N}=1$ Heterotic theory to M and F theories, as opposed to examples in the literature on only $\mathcal{N}=1$ Heteorotic/type IIA or

Heterotic/M-theory or Heterotic/F-theory dual pairs. In Section **2**, we propose that the M-theory side is given by a 7-manifold of $SU(3)\times\mathbf{Z}_2$-holonomy of the type $(CY\times S^1)/g.\mathcal{I}$, where g is a suitably defined freely-acting antiholomorphic involution on the CY which is precisely the same as the one considered in [1], Ω is the world-sheet parity and $\mathcal{I}$ reflects the S^1. These 7-manifolds are referred to as "barely G_2 manifolds" in [2]. In addition, the $D=4,\ \mathcal{N}=1$ Heterotic/F-theory dual models constructed have the following in common (as a consequence of applying fiberwise duality to Heterotic on T^2 being dual to F-theory on $K3$). The Heterotic theory is compactified on a CY_3 that is elliptically fibered over a 2-manifold B_2. The F-theory dual of this Heterotic theory is constructed by considering an elliptically fibered Calabi-Yau 4-fold X_4 that is elliptically fibered over a 3-manifold B_3. Additionally, the base B_3 is a $\mathbf{CP}^1$-fibration over B_2 (the same one that figures on the heterotic side). In Section **3**, we propose that the required Calabi-Yau 4-fold on the F-theory side is elliptically fibered over a trivially rationally ruled base given by $\mathbf{CP}^1\times\mathcal{E}$, $\mathcal{E}$ being the Enriques surface. Section **4** has a discussion on the results obtained and outlook for future work.

2 M-Theory Dual

In this section we construct the M-theory uplift of type IIA background of [1]. Now, the specific $\mathcal{N}=1$ Heterotic/type IIA dual pair of [1] that we will be considering in this letter is Heterotic on a CY given by $\frac{K3\times T^2}{\mathbf{Z}_2^E}$ and type IIA on orientifolds of CY's (the mirrors of which are) given as hypersurface of degree 24 in $\mathbf{WCP}^4[1,1,2,8,12]$, the mirror duals to which, are given by:

$$z_1^{24}+z_2^{24}+z_3^{12}+z_4^3+z_5^2-12\alpha z_1z_2z_3z_4z_5-2\beta z_1^6z_2^6z_3^6-\gamma z_1^{12}z_2^{12}=0. \tag{1}$$

The $\mathbf{Z}_2^E$ represents the Enriques involution times reflection of the T^2 as considered in [11, 3] given by the action

$$\mathbf{Z}_2^E:(u_1,u_2,u_3)\rightarrow(-u_1+\frac{1}{2},u_2+\frac{1}{2},-u_3), \tag{2}$$

($u_{1,2}$ and u_3 are coordinates on $K3$ and T^2 respectively) and the space-time orientation reversing antiholomorphic involution used for constructing the

CY orientifold is:

$$\omega : (z_1, z_2, z_3, z_4, z_5) \to (\bar{z}_2, -\bar{z}_1, \bar{z}_3, \bar{z}_4, \bar{z}_5). \tag{3}$$

Now, (barely) G_2-Manifolds of the type $\frac{CICY \times S^1}{g.I}$ where g was an antiholomorphic involution, were considered in [13] where g was one of the following, the former corresponding to an involution with fixed points, leading to G_2-holonomy manifolds and the latter acting freely leading to 'barely G_2-manifolds":

$$\begin{aligned}&(z_1, ..., z_{n+1}) \to (\bar{z}_1, ..., \bar{z}_{n+1}) \text{ for } \mathbf{CP}^{n+1}, \text{ or}\\ &(z_1, z_2..., z_{2n}) \to (\bar{z}_2, -\bar{z}_1, ..., \bar{z}_{2n}, -\bar{z}_{2n-1}) \text{ for } \mathbf{CP}^{2n-1},\end{aligned} \tag{4}$$

(for CICY expressed as a set of homogenous equations in a single $\mathbf{CP}^m$ space) where in the second choice, it is understood that the antiholomorphic involution acts pairwise on the homogenous coordinates. The antiholomorphic involution that we require is ω of (3), which is a combination of the ones in (4).[2] Another point worth keeping in mind is that under ω of (3), the Kähler form J going over to -J is only a statement in the cohomology group $H^{1,1}$. One can define inhomogenous coordinates for, e.g., Y, in the $z_2 \neq 0$ coordinate patch:

$$u \equiv \frac{z_1}{z_2};\ v \equiv \frac{z_3}{z_2^2};\ w \equiv \frac{z_4}{z_2^4}, \tag{5}$$

[using which one can solve for $\frac{z_5}{z_2^{12}}$ from the defining equation (1), and hence is not included as part of the CY coordinate system]. Then, one can show that

$$J \xrightarrow{\omega} -J + \mathcal{O}\Big(\frac{1}{|u|^{m>0}} \text{ or } g_{u\bar{u}} - \text{independent terms}\Big), \tag{6}$$

such that the $-J$ and $-J+$ extra terms both belong to the same cohomology class of $H^{1,1}$. As $u \in \mathbf{CP}^1$-base coordinate and $g_{u\bar{u}}$ gives the size of the $\mathbf{CP}^1$ base, in the large base-limit of [1], J under the antiholomorphic involution ω goes over to $-J$ **exactly**. Similarly, $H^{2,1}$ goes over to $H^{1,2}$ (and $X_{2,1} \in H^{2,1}$ goes over to $X_{1,2} \in H^{1,2}$ exactly in the large-base limit of [1]) but an element $Y^{1,1}$ of $H^{1,1}$ goes over to an element of the cohomology class $[-Y^{1,1}]$ of $H^{1,1}$

[2] To see that ω is an involution, one sees that ω^2 takes $(z_1, z_2, z_3, z_4, z_5)$ to $(-z_1, -z_2, z_3, z_4, z_5)$, which is projectively equal to unity as can be seen by setting the C^* variable t to -1 in $z_i \sim t^{w_i} z_i$ in $\mathbf{WCP}^4$ homogenous coordinates.

and no statement can be made for large base-limit exactness like the ones for J or Ω above. [3] To summarize, we get:

$$
\begin{aligned}
&[\omega^*(J)] = [-J];\ \omega^*(J) \stackrel{\text{large } \mathbf{CP}^1}{\longrightarrow} -J,\\
&[\omega^*(X)] = [\bar{X}];\ \omega^*(X) \stackrel{\text{large } \mathbf{CP}^1}{\longrightarrow} \bar{X};\\
&[\omega^*(Y)] = [-Y],
\end{aligned}
\tag{11}
$$

[3]The exact expressions for J and an element of $H^{2,1}$ under the action of ω, written in terms of inhomogenous coordinates in the $z_2 \neq 0$ coordinate patch are as follows.

$$
J = g_{u\bar{u}} du\wedge d\bar{u} + g_{v\bar{v}} dv\wedge d\bar{v} + g_{w\bar{w}} dw\wedge d\bar{w} + 2i\text{Im}\Big[g_{u\bar{v}} du\wedge d\bar{v} + g_{u\bar{w}} du\wedge d\bar{w} + g_{v\bar{w}} dv\wedge d\bar{w}\Big], \tag{7}
$$

under (3), which in terms of the CY coordinates (u,v,w) is written as $\omega : (u,v,w) \rightarrow (-\frac{1}{\bar{u}}, \frac{\bar{v}}{\bar{u}^2}, \frac{\bar{w}}{\bar{u}^2})$, goes over to

$$
\begin{aligned}
&-J + d\bar{u}\wedge du\Bigg(\frac{4|v|^2}{|u|^2} g_{v\bar{v}} + \frac{64|w|^2}{|u|^2} g_{w\bar{w}} + 2\text{Re}\Bigg[-\frac{2v}{u} g_{v\bar{u}} - \frac{8w}{u} g_{w\bar{u}} + \frac{16\bar{v}w}{|u|^2} g_{w\bar{v}}\Bigg]\Bigg)\\
&-4i\text{Im}\Bigg(\Bigg[\frac{v}{u} g_{v\bar{v}} + 4\frac{w}{u} g_{w\bar{u}}\Bigg] d\bar{v}\wedge du\Bigg) + 2i\text{Im}\Bigg(g_{w\bar{v}} d\bar{v}\wedge dw\Bigg)\\
&+2i\text{Im}\Bigg(\Bigg[-\frac{8\bar{w}}{\bar{u}} g_{w\bar{w}} + g_{w\bar{u}} - \frac{\bar{v}}{\bar{u}} g_{w\bar{v}}\Bigg] d\bar{u}\wedge dw\Bigg) \stackrel{|u|\to\infty,\ \frac{g_{u_i\bar{u}_j}}{g_{u\bar{u}}}\to 0, i,j\neq 1}{\longrightarrow} -J.
\end{aligned}
\tag{8}
$$

Consider $A \in H^{2,1}(CY_3)$, written out in components as:

$$
A = \sum_{i\neq j\neq k=1}^{3} A_{u_u u_j \bar{u}_k} du_i \wedge du_j \wedge d\bar{u}_k + \sum_{i\neq j=1}^{3} A_{u_i u_j \bar{u}_i} du_i \wedge du_j \wedge d\bar{u}_i, \tag{9}
$$

where $u_{1,2,3} \equiv u, v, w$. Then,

$$
\begin{aligned}
&A \stackrel{\omega}{\to} A^* + d\bar{u}\wedge d\bar{v}\wedge du\Bigg(\frac{2v}{u} A_{\bar{v}\bar{u}v} - \frac{16v\bar{w}}{|u|^2} A_{\bar{v}\bar{w}v} + \frac{64|w|^2}{|u|^2} A_{\bar{w}v\bar{v}w} - \frac{8w}{u} A_{\bar{u}\bar{v}w} + \frac{8\bar{w}}{\bar{u}} A_{\bar{v}\bar{w}u}\Bigg)\\
&+d\bar{u}\wedge d\bar{w}\wedge du\Bigg(\frac{2|v|^2}{|u|^2} A_{\bar{v}\bar{w}v} + \frac{8w}{u} A_{\bar{w}\bar{u}w} - \frac{16\bar{v}w}{|u|^2} A_{\bar{w}\bar{v}w} - \frac{2\bar{v}}{\bar{u}} A_{\bar{v}\bar{w}u} + \frac{2v}{u} A_{\bar{w}\bar{u}v}\Bigg)\\
&+d\bar{v}\wedge d\bar{u}\wedge dv(-\frac{8\bar{w}}{\bar{u}} A_{\bar{v}\bar{w}v}) + d\bar{w}\wedge d\bar{u}\wedge dw(-\frac{2\bar{v}}{\bar{u}} A_{\bar{w}\bar{v}w}) + d\bar{v}\wedge d\bar{w}\wedge du(-\frac{2v}{u} A_{\bar{v}\bar{w}v} + \frac{8w}{u} A_{\bar{w}\bar{v}w})\\
&+d\bar{u}\wedge d\bar{w}\wedge dv(-\frac{2\bar{v}}{\bar{u}} A_{\bar{v}\bar{w}v}) + d\bar{u}\wedge d\bar{v}\wedge dw(-\frac{8\bar{w}}{\bar{u}} A_{\bar{w}\bar{v}w}) \stackrel{|u|\to\infty}{\to} A^*.
\end{aligned}
\tag{10}
$$

where $X \in H^{2,1}(CY_3 \longrightarrow_{K3} \mathbf{CP}^1)$ and $Y \in H^{1,1}(CY_3 \longrightarrow_{K3} \mathbf{CP}^1)$, and [ ] denotes the cohomology class. The closed and co-closed calibration 3-form ϕ is given by:

$$\phi = J \wedge dx + Re(e^{-\frac{i\theta}{2}}\Omega), \tag{12}$$

where x is the S^1 coordinate, and Ω is the holomorphic 3-form of the $CY_3(3,243)$. To get the spectrum for M-theory compactified on the 'barely G_2 manifold' $\mathcal{Z} \equiv \frac{CY \times S^1}{\omega.\mathcal{I}}$, one sees (See [2]) that $\frac{1}{2}(H^{3,0}(CY) + H^{0,3}(CY))$ corresponding to $\frac{1}{2}(h^{3,0}(CY) + h^{0,3}(CY)) = 1$, is invariant under the $\mathbf{Z}_2$ involution ω. Similarly, $\frac{1}{2}(H^{2,1}(CY) + H^{1,2}(CY))$ corresponding to $\frac{1}{2}(h^{1,2}(CY) + h^{2,1}(CY)) = h^{2,1}(CY)$ elements, is invariant under the involution ω. As shown in [1], ω acts as -1 on $H^{1,1}(CY)$ implying that $H^{1,1}_+(CY)$, i.e., the part of $H^{1,1}(CY)$ even under ω is zero, and the part odd, $H^{1,1}_-(CY) = H^{1,1}(CY)$. Hence, n_V, n_C that denote respectively the number of vector and hypermultiplets, will be given by:

$$\begin{aligned} n_V(\mathcal{Z}) &= h^{1,1}_+(CY) = 0, \\ n_C(\mathcal{Z}) &= h^{2,1}(CY) + h^{3,0}(CY) + h^{1,1}_-(CY) = 243 + 1 + 3 = 247. \end{aligned} \tag{13}$$

M-theory on barely G_2 manifolds yielding $n_V = 0$ have also been considered in [14]. Lets now briefly review the spectrum of heterotic theory on $\frac{K3 \times T^2}{\mathbf{Z}_2}$ (See [1]), where the $\mathbf{Z}_2$-involution is the Enriques involution on the $K3$ and a reflection on the T^2. Because of the reflection on T^2, the four $\mathcal{N} = 2$ vector multiplets get projected out. The S, T, U moduli survive the involution as $\mathcal{N} = 1$ chiral multiplets. In addition, one gets 20 $\mathcal{N} = 1$ chiral multiplets from the Enriques surface moduli and 224 from the $E_8 \times E_8$ gauge bundle moduli (assuming that interchange of the two E_8 factors under the action of ω)[4] . Thus one gets a total of 3+20+224=247 $\mathcal{N} = 1$ chiral multipets.

Now coming to the spectrum of type IIA theory compactified on $\frac{CY}{\omega.\Omega}$ (See [1]). Denoting the CY indices by $i, \bar{j}$ and the four-dimensional indices by μ, the $\mathcal{N} = 2$ spectrum consist of $h^{1,1}(CY)$ vector multiplets the scalar components of each of which consists $(g_{i\bar{j}}, b_{i\bar{j}}, c_{\mu i\bar{j}})$ ($g_{i\bar{j}}$ from the $D = 10$ metric g_{MN} and $b_{i\bar{j}}$ from the $D = 10$ antisymmetric tensor B_{MN} from the NS-NS

[4]One gets 224 $\mathcal{N} = 2$ hypermultiplets by noting that the dual coexter for E_8, $C_2(E_8) = 30$ and using that the total number of neutral hypermultiplets coming from a single E_8 factor with $\int_{K3} c_2 = 12$ (for (12,12) instantons embedded in the two E_8's) is given by $30 \times 12 - 248 = 112$. Of each $\mathcal{N} = 2$ hypermultiplet, one $\mathcal{N} = 1$ chiral multiplet survives the involution.

sector, and $c_{\mu i\bar{j}}$ from the $D = 10$ 3-form gauge potential c_{MNP} from the RR sector), and $h^{2,1}(CY)$ hypermultiplets the bosonic components of each of which is given by $(g_{[ij]}, g_{[\bar{i}\bar{j}]}, c_{ij\bar{k}}, c_{\bar{i}\bar{j}k})$ ($g_{[ij]}$ and its complex conjugate from $g^{l\bar{l}}g^{k\bar{k}}X_{[i|\bar{l}\bar{k}}\Omega_{j]lk}$, $X_{1,2} \in H^{1,2}$ and $\Omega_{3,0}$ being the nowhere vanishing holomorphic 3-form)+ another hypermultiplet whose bosonic components are given by $(\phi, a(\sim b_{\mu\nu}), c_{ijk}, c_{\bar{i},\bar{j},\bar{k}})$. Among the $h^{1,1}(CY)$ $\mathcal{N} = 2$ vector multiplets, the gauge field gets projected out under $\omega.\Omega$ to give $h^{1,1}(CY)$ $\mathcal{N} = 1$ chiral multiplets. From the $h^{2,1}(CY) + 1$ $\mathcal{N} = 2$ hypermultiplets, one each from the two NS-NS and RR states survive the orientifold projection, yielding $h^{2,1}(CY) + 1$ $\mathcal{N} = 1$ chiral multiplets. Hence, one gets a total of $h^{1,1}(CY) + h^{2,1}(CY) + 13 + 243 + 1 = 247$ $\mathcal{N} = 1$ chiral multiplets

This one sees that the spectra associated with Heterotic on $\frac{K3\times T^2}{\mathbf{Z}_2}$, type IIA on $\frac{CY}{\omega.\Omega}$, and M-theory on $\frac{CY\times S^1}{\omega.\mathcal{I}}$ match.

3 F-Theory Dual

We now show the possibility of finding an $\mathcal{N} = 1$ triality between the $\mathcal{N} = 1$ heterotic on $CY_3(11, 11)$(/type IIA on $\frac{CY_3(3,243)}{\omega.\Omega}$ dual pair) of Vafa-Witten, M theory on the “barely G_2 manifold” $\frac{CY_3(3,243)\times S^1}{\omega.\mathcal{I}}$ of $SU(3)\times\mathbf{Z}_2$ holonomy, and F-theory on an elliptically fibered X_4 , where the “11,11” and “3,243” denote the Hodge numbers, ω is an orienation-reversing antiholomorphic involution, $\mathcal{I}$ reverses the S^1. In principle, one should be able to get the right $\mathcal{N} = 1$ F-theory model by following the $\mathcal{N} = 2$ Higgs chain of [12]: $E_8 \times E_8 \rightarrow E_7\times E_7 \rightarrow E_6\times E_6 \rightarrow SO(10)\times SO(10) \rightarrow SO(9)\times SO(9) \rightarrow SO(8)\times SO(8) \rightarrow SO(7)\times SO(7) \rightarrow SU(4)\times SU(4) \rightarrow SU(3)\times SU(3) \rightarrow SU(2)\times SU(2) \rightarrow SU(1) \times SU(1)$ by embedding of suitable gauge commutants on the $K3$, or equivalently by embedding of $E_8 \times E_8$ vector bundle on $K3$ in one step, followed by tensoring with a T^2 and taking a suitable $\mathbf{Z}_2$-involution. One uses the Kodaira classification of singularities to count the F-theory moduli. We however do not follow this approach in the following.

The CY_3 on the heterotic side that we are interested in is one that is obtained by a freely-acting Enriques involution acting on the $K3$ times a reflection of the T^2, in $K3 \times T^2$, i.e., the Voisin-Borcea elliptically fibered $CY_3(11, 11) \equiv \frac{K3\times T^2}{g.\mathcal{I}}$, where g is the generator of the Enriques involution and $\mathcal{I}$ reflects the T^2. Hence, the B_2 above is $\frac{K3}{g}$. Now, the $\mathcal{N} = 2$ dual pair in [10] consisted of embedding $SU(2) \times SU(2)$ in $E_8 \times E_8$ on the Heterotic side, resulting in $E_7 \times E_7$, which is then Higgsed away. All that survives from the T^2 in $K3 \times T^2$ are the abelian gauge fields corresponding to $U(1)^4$. As shown in Vafa-Witten's paper[1], in the $\mathcal{N} = 1$ dual pair obtained by suitable $\mathbf{Z}_2$-moddings of both sides of the $\mathcal{N} = 2$ Heterotic/type IIA dual pair, the $U(1)^4$ gets projected out so that there are no vector multiplets and one gets 247 $\mathcal{N} = 1$ chiral multiplets on the Heterotic side on $CY_3(11, 11)$. We should be able to get the same spectrum on the F-theory side. If r denotes the rank of the unbroken gauge group in Heterotic theory, then the number of $\mathcal{N} = 1$ chiral multiplets in F-theory is given by the formula:

$$n_C = \frac{\chi(X_4)}{6} - 10 + h^{2,1}(X_4) - r, \tag{14}$$

which excludes the S modulus of the Heterotic theory. The rank r in turn is expressed as:

$$r = h^{1,1}(X_4) - h^{1,1}(B_3) - 1 + h^{2,1}(B_3). \tag{15}$$

For Heterotic theory on $CY_3(11, 11)$, $r = 0$.

The fibration structure can be summarized as: $X_4 \longrightarrow_{T^2} B_3 \longrightarrow_{\mathbf{CP}^1} B_2 \equiv \frac{K3}{g} \equiv \mathcal{E} \equiv$Enriques surface. Given that for elliptically fibered X_4, $h^{1,1}(X_4) - h^{1,1}(B_3) - 1 \geq 0$, $r = 0$ implies that

$$\begin{aligned} &h^{1,1}(X_4) = h^{1,1}(B_3) + 1 > 0; \\ &h^{2,1}(B_3) = 0. \end{aligned} \tag{16}$$

Now, in [15], $r = 0$ was obtained by embedding an $E_8 \times E_8$ vector bundle for which it was shown that the number of space-time filling F-theory 3 branes, needed for tadpole cacellation, matched the number of Heterotic 5-branes needed for anomaly cancellation. In [16] the brane match was shown for the case of embedding an $SU(n_1) \times SU(n_2)$ vector bundle in $E_8 \times E_8$, resulting in some unbroken gauge group. The difference in our situation is that unlike [16], for the $\mathcal{N} = 1$ model of [1], at the $\mathcal{N} = 2$ level, one has to embed an $SU(2) \times SU(2)$ in the $E_8 \times E_8$ on the $K3(\times T^2)$, a Calabi-Yau 2-fold($\times T^2$), and the resulting $E_7 \times E_7$ is Higgsed away, or equivalently, an $E_8 \times E_8$ vector bundle is embedded in $E_8 \times E_8$ on the Calabi-Yau 2-fold and not a Calabi-Yau 3-fold. The $\mathcal{N} = 1$ Heterotic model on the Voisin-Borcea Calabi-Yau 3-fold with Hodge numbers 11,11 is then obtained by a suitable $\mathbf{Z}_2$ modding of the $\mathcal{N} = 2$ model. Hence, it is not that one gets an $\mathcal{N} = 1$ model by embedding a gauge bundle on a Calabi-Yau 3-fold Z (that may or may not be followed by a Higgsing), but one gets the required $\mathcal{N}=1$ model as a three-step process: embedding a gauge bundle on a Calabi-Yau 2-fold times T^2, followed by complete Higgsing away of the resultant gauge group (the embedding and Higgsing can be combined into a single step of suitable embedding as discussed above), and finally modding by a (freely acting) involution yielding a Calabi-Yau 3-fold Z. We can still write that the total number of Heterotic moduli is given by the expression:

$$N_{het} = h^{1,1}(Z) + h^{2,1}(Z) + n_{bundle}, \tag{17}$$

where the bundle moduli correspond to an involution τ which acts trivially on the base and as reflection of the fiber (that can always be defined on an elliptically fibered Z [15]). It no longer can be defined as $h^1(Z, ad(V)) = I + 2n_o$, where the character-valued index I is given by -$\sum_{i=}^{3}(-)^i Tr_{H^i(Z,Ad(V))}(\frac{1+\tau}{2}) = -\sum_{i=0}^{3}(-)^i h^i_e(Z, Ad(V)) = n_e - n_o$ for no unbroken $\mathcal{N}=1$ gauge group, and e, o referring to even,odd respectively under the involution τ. However, given that such an involution τ exists, one can still write that

$$n_{bundle} = n_e + n_o = \mathcal{I} + 2n_o, \tag{18}$$

for a suitable "index" $\mathcal{I}$. We assume that at the τ-invariant point, the action of τ can be lifted to an action of the gauge bundle embedded at the level of $K3$. This index will encode the information about $I(K3, Ad(SU(2)\times SU(2)))$ and the Higgsing away of the $E_7\times E_7$, or equivalently $I(K3, Ad(E_8\times E_8))$ at the $\mathcal{N}=2$ level, and the freely acting Enriques involution times reflection of T^2. [5] There are no non-perturbative Heterotic 5-branes in the $\mathcal{N}=1$ model of [1]. Hence, for the $\mathcal{N}=1$ Heterotic/F-theory duality to hold, there will no F-theory 3-branes(given by $\frac{\chi(X_4)}{24}$) either, which implies that the elliptically fibered Calabi-Yau 4-fold must satisfy the constraint:

$$\chi(X_4) = 0. \tag{19}$$

Now,

$$\begin{aligned} h^{1,1}\Big(CY_3(11,11)\Big) &= 11 - \int_{\mathcal{E}} c_1^2(\mathcal{E}) = 11, \\ h^{2,1}\Big(CY_3(11,11)\Big) &= 11 + 29\int_{\mathcal{E}} c_1^2(\mathcal{E}) = 11, \end{aligned} \tag{20}$$

[5] As A.Klemm pointed out to us, in general, one can always write the index $\mathcal{I}$ as $a + b\int_{\mathcal{E}} c_1^2(\mathcal{E}) + c\int_{\mathcal{E}} c_2(\mathcal{E}) + d\int_{\mathcal{E}} c_1^2(\mathcal{T}) + e\int_{\mathcal{E}} c_2(\mathcal{T}) + f\int_{\mathcal{E}} c_1(\mathcal{E})\wedge c_1(\mathcal{T})$, where a, b, c, d, e, f are constants and $\mathcal{T}$ is a line bundle over $\mathcal{E}$.

as $c_1^2(\mathcal{E})=0$. Further, $\int_{\mathcal{E}} c_2(\mathcal{E})=12$. Assuming only a single section of the elliptic fibration: $Z \rightarrow_{T^2} \mathcal{E}(\equiv$ Enriques surface) and no 4-flux, from general considerations (See [16]), the Hodge data of X_4 will be given by:

$$\begin{aligned}
&h^{1,1}(X_4)=h^{1,1}(Z)+1+r=12-\int_{\mathcal{E}} c_1^2(\mathcal{E})+r, \\
&h^{2,1}(X_4)=n_o, \\
&h^{3,1}(X_4)=h^{2,1}+\mathcal{I}+n_o+1=12+29\int_{\mathcal{E}} c_1^2(\mathcal{E})+\mathcal{I}+h^{2,1}(X_4). \quad (21)
\end{aligned}$$

Now $t \equiv c_1(\mathcal{T})$ ($\mathcal{T}$ being a line bundle over B_2), the analog of n in the Hirzebruch surface F_n, is a measure of the non-triviality of the $\mathbf{CP}^1$-fibration of the rationally ruled B_3. Now, the $CY_3(3,243)$ on the type IIA side, can be represented as elliptic fibration over the Hirzebruch surface F_n, where n denotes the non-triviality of fibration of $\mathbf{CP}^1_f$ over $\mathbf{CP}^1_b$. The Weierstrass equation for $n=0$ is given by:

$$y^2=x^3+\sum_{i=0}^{8} f_i^{(8)}(z_1)z_2^i+\sum_{i=0}^{12} g_i^{(12)}(z_1)z_2^i, \quad (22)$$

implying that the number of complex structure deformations, $h^{2,1}$ is given by $9\times 9+13\times 13-(3+3+1)=243$. [6] Hence, analogous to setting $n=0$, we can set $t=0$ and doing so would imply the triviality of the fibration: $B_3=\mathbf{CP}^1\times B_2=\mathbf{CP}^1\times\mathcal{E}$, for which $h^{2,1}(B_3)=0$ thereby satisfying (16).

Equating n_{het} to 246, one gets from (17) and (18) the following

$$\mathcal{I}+2n_o=224. \quad (24)$$

There are no vector multiplets, and in addition to the heterotic dilaton, n_{het} has to correspond to the number of $\mathcal{N}=1$ chiral multiplets n_C on the F-theory side. Given that $r=\chi(X_4)=0$, from (14) one gets:

$$h^{2,1}(X_4)=128=n_o. \quad (25)$$

[6] Interestingly, for $n=2$, the Weierstrass equation is given by:

$$y^2=x^3+\sum_{i=-4}^{4} f_{8-4i}(z_1)z_2^{4-i}x+\sum_{i=-8}^{8} g_{12-2i}(z_1)z_2^{8-i}, \quad (23)$$

implying that the number of complex structure deformations, $h^{2,1}$ is given by $(17+15+13+...+3+1=)81+(25+23+...+3+1=)169-(3+3+1)=243$. Hence, elliptic fibrations over both F_0 and F_2 give the same hodge numbers. We will work with F_0.

This using (24), one gets

$$\mathcal{I} = -32. \tag{26}$$

Using the relation from [17]:

$$\frac{\chi(X_4)}{6} = 8 + h^{1,1}(X_4) - h^{2,1}(X_4) + h^{3,1}(X_4), \tag{27}$$

one sees that the elliptically 4-manifold X_4 that we are looking for is characterized by:

$$\begin{aligned} &h^{1,1}(X_4) = 12, \\ &h^{2,1}(X_4) = 128, \\ &h^{3,1}(X_4) = 108. \end{aligned} \tag{28}$$

This is consistent with (21). The $h^{2,2}(X_4)$ can be determined by the following relation [18]

$$h^{2,2}(X_4) = 2(22 + 2h^{1,1}(X_4) + 2h^{3,1}(X_4) - h^{2,1}(X_4)) = 268, \tag{29}$$

which has been obtained from the definitions of elliptic genera in terms of hodge numbers and as integrals involving suitable powers of suitable Chern classes, and $c_1(X_4) = 0$. Hence, $\mathcal{N} = 1$ Heterotic Theory on $\frac{K3\times T^2}{\mathbf{Z_2}}$ is dual to F-theory on an elliptically fibered Calabi-Yau 4-fold: $X_4[h^{1,1} = 12, h^{2,1} = 128, h^{3,1} = 108; 0] \rightarrow_{T^2} \mathbf{CP}^1 \times \mathcal{E}$. At the $\mathcal{N} = 2$ level, Heterotic on $K3 \times T^2$ should be dual to F-theory on $CY_3(3, 243) \times T^2$ as a consequence of repeated fiberwise application of duality to the basic duality that Heterotic on T^2 is dual to F-theory on $K3$, as well as because type IIA on a CY_3 should be dual to F-theory on $CY_3 \times T^2$ and Heterotic on $K3 \times T^2$ is dual to type IIA on $CY_3(3, 243)$. Hence, it is possible that an orbifold of $K3\times T^2$ on the Heterotic side should correspond to a suitable orbifold of $CY_3\times T^2$ on the F-theory side. Note, however, even though a naive freely acting orbifold of $CY_3(3, 243)\times T^2$ gives the right null Euler Characteristic, it can not, for instance, give $h^{1,1} = 12$, i.e., an enhancement over the $h^{1,1}(CY_3(3, 243) \times T^2) = 3 + 1 = 4$. This is unlike the case of the F-theory dual of Heterotic on Voisin-Borcea $CY_3(19, 19)$ which corresponded to an involution with fixed points, considered in [19].

The CY_4 with the required fibration structure and Hodge data given in (28) and (29) is missing from the list of hypersurfaces in $\mathbf{WCP}^5$ of Kreuzer and Skarke because it is not possible to get the desired CY_4 as a hypersurface in any toric variety as fibrations of toric hypersurfaces have bases that are toric varieties, and the Enriques surface, $\mathcal{E}$, is not a toric variety. Perhaps, one needs a "nef partition" (one could use "nef.x" part of the package PALP[20]) that makes the base, $\mathbf{CP}^1 \times \mathcal{E}$ a toric hypersurface. One might have to work with complete intersections in toric varieties.[7]

4 Conclusion

In this paper, we relate the $\mathcal{N} = 1$ Heterotic theory on a self-mirror CY_3 to the nonperturbative formulations of type IIA and IIB, namely M and F theories. While on the M-theory side, the suitable manifold turned out to one of $SU(3) \times \mathbf{Z}_2$ holonomy, referred to as a 'barely G_2 manifold', the elliptically fibered Calabi-Yau 4-fold involves a trivial $\mathbf{CP}^1$-fibration over the Enriques surface for its base, and surprisingly has a Hodge data that can not be obtained as a free involution of ($\mathcal{N} = 2$ F-theory on) $CY_3(3, 243) \times T^2$. The orientation-reversing antiholomorphic involution used, both in constructing CY_3 orientifold on the type IIA side, as well as the barely G_2 manifold, has some interesting properties that become manifest in the large $\mathbf{CP}^1$-base limit of $CY_3(3, 243)$ which we also discuss.

The precise construction of the CY_4 used in the F-theory dual and its connection with the $\mathcal{N} = 2$ parent model of F-theory on $CY_3(3, 243) \times T^2$, needs to be understood. It will also be interesting to calculate and compare the superpotentials on all sides.

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